ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 158 - Calculus for Engineering II

Final Examination

26.08.2022

Student Number:

Name-Surname:

Signature:

Duration: 100 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
6		20
Total		120

IMPORTANT NOTES:

- Check that the exam paper contains 6 questions.
- **Show all steps of your work.** Both the correct method and correct result are necessary to get full point.

1) Evaluate the following limits (if they exist):

a) (10 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+4y^2}{x^2+y^2}$

b) (10 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^6+y^4}$

2) This question has three independent parts.

a) (6 points) Let $f(x,y) = \dfrac{x \sin x}{1+e^x} - 3x^3y^4 + 12 \ln y$. Find f_{yy}.

b) (7 points) Let $y^2 z \ln x + 4x^3 y - (1+x^2)e^{z-4} = 6$.

Find the value of z_x at $x = 1,\ y = 2,\ z = 4$.

c) (7 points) Let $f = u^2 + v^3, \quad u = 2xy,\ v = x^2 + y^2$.

Find the value of $\dfrac{\partial f}{\partial x}$ at $x = 3,\ y = 1$.

3) Find and classify the critical points of the function:

$$f(x, y) = 6x^3 + 2xy^2 - 18x + 12$$

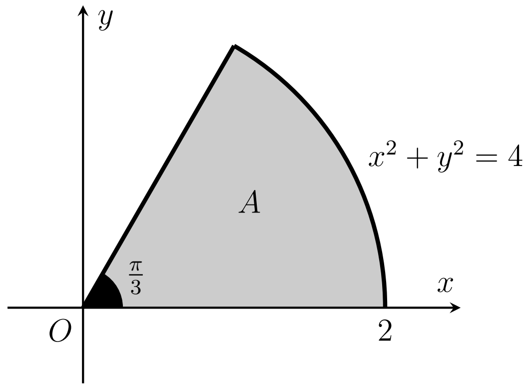
4) Find the maximum and minimum values of the function $f(x, y) = xy + 18$ subject to the constraint $x^2 + 4y^2 = 16$.

5) Evaluate the following the double integrals:

a) (10 points) $\displaystyle\int_1^3 \int_0^2 \frac{2xy^3}{1+x^2}\, dy\, dx$

b) (10 points) $\displaystyle\int_0^8 \int_{x/4}^2 e^{y^2}\, dy\, dx$

6) Evaluate the integral $\displaystyle\iint_A \sqrt{x^2+y^2}\,dy\,dx$ where A is the region indicated in the figure:



Final Exam – Solutions

1) a) Along $y=0$: $\lim_{x\to 0} \frac{2x^3}{x^2} = \lim_{x\to 0} 2x = 0$

Along $x=0$: $\lim_{y\to 0} \frac{4y^2}{y^2} = \lim_{y\to 0} 4 = 4$

limits are diff $\Rightarrow$ By Two path test limit dne

or

$$\lim_{r\to 0} \frac{2r^3\cos^3\theta + 4r^2\sin^2\theta}{r^2} = \lim_{r\to 0} 2r\cos^3\theta + 4\sin^2\theta = 4\sin^2\theta$$

limit depends on θ $\Rightarrow$ Limit dne.

b) $0 \le y^4 \le y^4 + x^6 \Rightarrow 0 \le \frac{y^4}{x^6+y^4} \le 1 \Rightarrow 0 \le \frac{x^2y^4}{x^6+y^4} \le x^2$

$0 \to 0$, $x^2 \to 0$ as $(x,y)\to(0,0)$

By sand. thm, $\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^6+y^4} = 0$

2) a) $f_y(x,y) = 0 - 12x^3y^3 + 12 \cdot \frac{1}{y}$

$$f_{yy} = -36x^3y^2 - \frac{12}{y^2}$$

b) By Implicit Diff:

$$F(x,y,z) = y^2 z \ln x + 4x^3 y - (1+x^2)\, e^{z-4}$$

$$z_x = -\frac{F_x}{F_z}, \quad F_x = \frac{y^2 z}{x} + 12x^2 y - 2x\, e^{z-4}$$

$$F_z = y^2 \ln x - (1+x^2)\, e^{z-4}$$

$$z_x = -\frac{\frac{y^2 z}{x} + 12x^2y - 2x\, e^{z-4}}{y^2 \ln x - (1+x^2)\, e^{z-4}} \Rightarrow z_x \Big|_{\substack{x=1 \\ y=2 \\ z=4}} = -\frac{16+24-2}{0-2} = +19$$

c) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2u\, 2y + 3v^2\, 2x$$

$$\frac{\partial f}{\partial x}\Big|_{x=3,\, y=1} = 2 \cdot 6 \cdot 2 \cdot 1 + 3 \cdot 10^2 \cdot 2 \cdot 3 = 1824$$

$\Rightarrow u = 6, v = 10$

3) $f(x,y) = 6x^3 + 2xy^2 - 18x + 12$

$f_x = 18x^2 + 2y^2 - 18 = 0$

$f_y = 4xy = 0 \Rightarrow x = 0$ or $y = 0$

$x = 0 \downarrow 2y^2 = 18 \downarrow y = \mp 3$

$y = 0 \downarrow 18x^2 = 18 \downarrow x = \mp 1$

$\Rightarrow (0,-3), (0,3), (-1,0), (1,0)$ are critical points

$A(x,y) = f_{xx} = 36x, \quad B(x,y) = f_{yy} = 4x, \quad C(x,y) = 4y$

	A	B	C	$\Delta = AB - C^2$	
(0,-3)	0	0	-12	$-144 < 0$	→ saddle
(0,3)	0	0	12	$-144 < 0$	→ saddle
(-1,0)	-36	-4	0	$144 > 0$	→ loc max
(1,0)	36	4	0	$144 > 0$	→ loc min

(0,-3) and (0,3) are saddle points

f has a loc min at (1,0), loc max at (-1,0).

4) $y = \lambda \cdot 2x$

$x = \lambda \cdot 8y$

$x^2 + 4y^2 = 16$

$\Rightarrow \lambda = \frac{y}{2x} = \frac{x}{8y} \Rightarrow x^2 = 4y^2$

$8y^2 = 16 \Rightarrow y = \pm\sqrt{2}, \quad x = \pm 2\sqrt{2}$

(x,y)	$f(x,y)$	
$(2\sqrt{2}, \sqrt{2})$	22	max
$(2\sqrt{2}, -\sqrt{2})$	14	min
$(-2\sqrt{2}, \sqrt{2})$	14	min
$(-2\sqrt{2}, -\sqrt{2})$	22	max

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5) a) $\int_1^3 \int_0^2 \frac{2xy^3}{1+x^2} \, dy \, dx$

$= \int_1^3 \frac{2x}{1+x^2} \left(\frac{y^4}{4}\right)\Big|_0^2 dx$

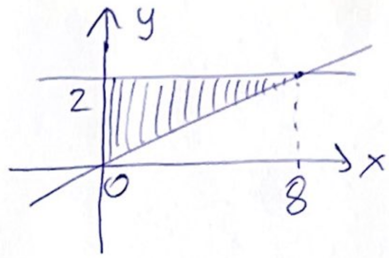
$= 4 \int_1^3 \frac{2x\,dx}{1+x^2}$

$= 4 \ln(1+x^2) \Big|_1^3$

$= 4(\ln 10 - \ln 2)$

$= 4 \ln 5$

5 b)



Reverse the Order:

$$\int_0^8 \int_{x/4}^2 e^{y^2}\, dy\, dx = \int_0^2 \int_0^{4y} e^{y^2}\, dx\, dy$$

$$= \int_0^2 4y e^{y^2}\, dy$$

$$= 2e^{y^2} \Big|_0^2$$

$$= 2(e^4 - 1)$$

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6) $$\iint_A \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/3} \int_0^2 r \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/3} \frac{r^3}{3} \Big|_0^2 \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/3} d\theta$$

$$= \frac{8}{3} \cdot \frac{\pi}{3}$$

$$= \frac{8\pi}{9}$$