ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 158 - Calculus for Engineering II

2021-2022 Summer

MIDTERM EXAMINATION

05.08.2022, 15:20

Question	Grade	Out of
1		15
2		20
3		15
4		15
5		20
6		15
Total		100

STUDENT NUMBER:

NAME-SURNAME: KEY

SIGNATURE:

DURATION: 100 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. a) (7 points) Is the sequence $a_n = \left(1 - \frac{2}{n}\right)^{-\frac{n}{3}}$ convergent or divergent?

Explain your answer.

$$L = \lim_{n\to\infty} a_n$$

$$\ln L = \lim_{n\to\infty} -\frac{n}{3} \ln\left(1-\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1-\frac{2}{n}\right)}{-\frac{3}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1-\frac{2}{n}} \cdot \frac{2}{n^2}}{\frac{3}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2}{3} \cdot \frac{1}{1-\frac{2}{n}}$$

$$= \frac{2}{3}$$

$$\Rightarrow L = e^{2/3}$$

Convergent

b) (8 points) Is the series $\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{-\frac{n}{3}}$ convergent or divergent?

Explain and indicate which tests you use.

$$\lim_{n\to\infty} a_n = e^{2/3} \neq 0$$

Divergent by n^{th} term test.

2. Are the following series convergent or divergent? Explain and indicate which tests you use.

a) (7 points) $\sum_{n=1}^{\infty} \frac{3^{-n}}{\ln(1+n^2)}$

$\sum 3^{-n} = \sum \left(\frac{1}{3}\right)^n$ is convergent by geometric series test.

$r = \frac{1}{3} < 1$

$\ln(1+n^2) > 1$ for $n \geq 2$ $\Rightarrow$ $\frac{\left(\frac{1}{3}\right)^n}{\ln(1+n^2)} < \left(\frac{1}{3}\right)^n$

$\Rightarrow \sum \frac{3^{-n}}{\ln(1+n^2)}$ is convergent by comparison test

b) (7 points) $\sum_{n=0}^{\infty} \frac{n^2-2}{2n^3+n^2+1}$

$\sum \frac{1}{n}$ is divergent by integral test. (Harmonic Series)

$$\lim_{n\to\infty} \frac{\frac{n^2-2}{2n^3+n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-2n}{2n^3+n^2+1} = \frac{1}{2}$$

Given series is divergent by Limit Comparison Test

c) (6 points) $\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}} = 1 \neq 0$$

$\Rightarrow \sum 2^{\frac{1}{n}}$ is divergent by n^{th} term test.

3. Are the following series convergent or divergent? Explain and indicate which tests you use.

a) (7 points) $\sum_{n=2}^{\infty} \frac{(n-1)!}{(n+1)!}(2n)!$

Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n!}{(n+2)!}(2n+2)!}{\frac{(n-1)!}{(n+1)!}(2n)!}$$

$$= \lim_{n\to\infty} \frac{n(2n+2)(2n+1)}{n+2}$$

$$= \infty$$

$\Rightarrow$ Divergent

n^{th} term test:

$$\lim_{n\to\infty} \frac{(n-1)!}{(n+1)!}(2n)!$$

$$= \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)!}{(n+1)n}$$

$$= \infty$$

$\Rightarrow$ Divergent

b) (8 points) $\sum_{n=1}^{\infty} \left(\frac{n\ln(n)}{1+n^2}\right)^n$

Root Test:

$$\lim_{n\to\infty} \left(\left(\frac{n\ln n}{1+n^2}\right)^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{n\ln n}{1+n^2} \quad \left(=\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\ln n + 1}{2n} \quad \left(=\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{2}$$

$$= 0 < 1 \Rightarrow \text{Convergent}$$

4. (15 points) Is the series $\sum_{n=1}^{\infty} \frac{n^2 \cos(n\pi)}{n^3+1}$ absolutely convergent, conditionally convergent or divergent?

$\cos(n\pi) = (-1)^n \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+1}$

• Abs Conv: $|a_n| = \left| (-1)^n \frac{n^2}{n^3+1} \right| = \frac{n^2}{n^3+1} \approx \frac{1}{n}$

$\lim_{n\to\infty} \frac{n^2/n^3+1}{1/n} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \Rightarrow 0 < 1 < \infty$

Since $\sum_{n=1}^{\infty} 1/n$ is div, by LCT $\sum |a_n|$ is also div.

• Cond Conv: $u_n = \frac{n^2}{n^3+1}$

• $u_n > 0 \ \forall n$

• $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$

• $f(x) = \frac{x^2}{x^3+1} \Rightarrow f'(x) = \frac{2x(x^3+1) - x^2 3x^2}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2} < 0$ for $x \geq 2$.

$\Rightarrow u_n = \frac{n^2}{n^3+1}$ is dec for $n \geq 2$.

By AST, it is conv

$\therefore$ Given series cond conv

5. a) (10 points) Find the Maclaurin series (Taylor series around $x = 0$) generated by $f(x) = \ln(1+2x)$.

$$f(x) = \ln(1+2x) \Rightarrow f'(x) = \frac{1}{1+2x} \cdot 2 = 2 \cdot \frac{1}{1-(-2x)}$$

$$\Rightarrow f'(x) = 2 \cdot \sum_{n=0}^{\infty} (-2x)^n \quad \text{for } -1 < -2x < 1$$

$$\Rightarrow f'(x) = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} x^n \quad \text{for } -1/2 < x < 1/2$$

$$\Rightarrow f(x) = \ln(1+2x) = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{x^{n+1}}{n+1} \quad \text{for } -1/2 < x < 1/2$$

b) (5 points) What is the <u>radius</u> and the <u>interval</u> of convergence?

$$x = -1/2 \Rightarrow \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{(-1/2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^{n+n+1} = \sum_{n=0}^{\infty} -1$$ by n-th term test, it is div.

$$x = 1/2 \Rightarrow \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{(1/2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$, $u_n = 1/n+1 > 0$, dec $\lim_{n\to\infty} u_n = 0$

by AST, it is conv

$$IC = (-1/2, 1/2] \quad , R = 1/2$$

c) (5 points) Estimate $\ln(1.8)$ using first two terms of the series you found in part a).

$$\ln(1.8) = \ln(1+0.8) = \ln(1+2 \cdot 0{,}4) \underset{x=0{,}4}{=} \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{(0.4)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \left(4/5\right)^{n+1}$$

$$\ln(1.8) \approx +\frac{4}{5} - \frac{1}{2}\frac{4^2}{5^2} = +\frac{4}{5} - \frac{16}{50} = +\frac{24}{50}$$

6. Equation of a line and two planes are given as

$$\ell: \quad x = 3t, \quad y = -1 + 3t, \quad z = -2 + 3t, \quad -\infty < t < \infty,$$

$$P: \quad 4x - y - z = 3, \quad \text{and} \quad Q: \quad x - y + 2z = 3.$$

a) (7 points) Find the line of intersection of the given planes.

- $\left.\begin{array}{l} n_P = \langle 4, -1, -1 \rangle \\ n_Q = \langle 1, -1, 2 \rangle \end{array}\right\} n_P \times n_Q = \begin{vmatrix} i & j & k \\ 4 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = -3i - 9j - 3k \neq 0 \Rightarrow n_P \not\parallel n_Q$

- A point on the line (on the P and Q): choose $x = 0$

$\left.\begin{array}{l} y + z = -3 \\ -y + 2z = 3 \end{array}\right\} z = 0 \Rightarrow y = -3 \Rightarrow P_0 = (x_0, y_0, z_0) = (0, -3, 0)$

- Direction vector: $\vec{u} \parallel n_P \times n_Q \Rightarrow \vec{u} = \langle -3, -9, -3 \rangle$

- Equation: $x = 0 - 3t$
  $y = -3 - 9t$
  $z = 0 - 3t, \quad t \in \mathbb{R}$

b) (8 points)

- If the line ℓ and the plane Q are parallel, find the distance between them.
- If they are not, find their point of intersection.

- $\left.\begin{array}{l} \vec{u} = \langle 3, 3, 3 \rangle \\ n_Q = \langle 1, -1, 2 \rangle \end{array}\right\} \begin{array}{l} \vec{u} \cdot \vec{n}_Q = \langle 3, 3, 3 \rangle \cdot \langle 1, -1, 2 \rangle = 3 - 3 + 6 = 6 \neq 0 \\ \vec{u} \not\perp n_Q \Rightarrow \ell \text{ is not parallel to } Q \end{array}$

- Intersection point:

$x = 3t, \; y = -1 + 3t, \; z = -2 + 3t \Rightarrow (3t) - (-1 + 3t) + 2(-2 + 3t) = 3$

$\Rightarrow 3t + 1 - 3t - 4 + 6t = 3$

$\Rightarrow 6t = 6 \Rightarrow t = 1$

Intersection point is $(3, 2, 1)$ and distance is 0.